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Moment estimates for Lévy Processes

HARALD LUSCHGY* AND GILLES PAGÈS†

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Abstract

For real Lévy processes $(X_t)_{t \geq 0}$ having no Brownian component with Blumenthal-Gettoor index β , the estimate $\mathbb{E} \sup_{s \leq t} |X_s - a_p s|^p \leq C_p t$ for every $t \in [0, 1]$ and suitable $a_p \in \mathbb{R}$ has been established by Millar [6] for $\beta < p \leq 2$ provided $X_1 \in L^p$. We derive extensions of these estimates to the cases $p > 2$ and $p \leq \beta$.

Key words: Lévy process increment, Lévy measure, α -stable process, Normal Inverse Gaussian process, tempered stable process, Meixner process.

2000 Mathematics Subject Classification: 60G51, 60G18.

1 Introduction and results

We investigate the L^p -norm (or quasi-norm) of the maximum process of real Lévy processes having no Brownian component. A (càdlàg) Lévy process $X = (X_t)_{t \geq 0}$ is characterized by its so-called local characteristics in the Lévy-Khintchine formula. They depend on the way the "big" jumps are truncated. We will adopt in the following the convention that the truncation occurs at size 1. So that

$$\mathbb{E} e^{iuX_t} = e^{-t\Psi(u)} \text{ with } \Psi(u) = -iua + \frac{1}{2}\sigma^2 u^2 - \int (e^{iux} - 1 - iux \mathbf{1}_{\{|x| \leq 1\}}) d\nu(x) \quad (1.1)$$

where $u, a \in \mathbb{R}, \sigma^2 \geq 0$ and ν is a measure on \mathbb{R} such that $\nu(\{0\}) = 0$ and $\int x^2 \wedge 1 d\nu(x) < +\infty$. The measure ν is called the Lévy measure of X and the quantities (a, σ^2, ν) are referred to as the characteristics of X . One shows that for $p > 0$, $\mathbb{E}|X_1|^p < +\infty$ if and only if $\mathbb{E}|X_t|^p < +\infty$ for every $t \geq 0$ and this in turn is equivalent to $\mathbb{E} \sup_{s \leq t} |X_s|^p < +\infty$ for every $t \geq 0$. Furthermore,

$$\mathbb{E}|X_1|^p < +\infty \text{ if and only if } \int_{\{|x| > 1\}} |x|^p d\nu(x) < +\infty \quad (1.2)$$

(see [7]). The index β of the process X introduced in [2] is defined by

$$\beta = \inf\{p > 0 : \int_{\{|x| \leq 1\}} |x|^p d\nu(x) < +\infty\}. \quad (1.3)$$

Necessarily, $\beta \in [0, 2]$.

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In the sequel we will assume that $\sigma^2 = 0$, i.e. that X has no Brownian component. Then the Lévy-It decomposition of X reads

$$X_t = at + \int_0^t \int_{\{|x| \leq 1\}} x(\mu - \lambda \otimes \nu)(ds, dx) + \int_0^t \int_{\{|x| > 1\}} x\mu(ds, dx) \quad (1.4)$$

where λ denotes the Lebesgue measure and μ is the Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}$ associated with the jumps of X by

$$\mu = \sum_{t \geq 0} \varepsilon_{(t, \Delta X_t)} \mathbf{1}_{\{\Delta X_t \neq 0\}},$$

$\Delta X_t = X_t - X_{t-}$, $\Delta X_0 = 0$ (see [4], [7]).

Theorem 1 *Let $(X_t)_{t \geq 0}$ be a Lévy process with characteristics $(a, 0, \nu)$ and index β such that $\mathbb{E}|X_1|^p < +\infty$ for some $p \in (\beta, \infty)$ or for $p = \beta$ provided $\int_{\{|x| \leq 1\}} |x|^\beta d\nu(x) < +\infty$ and $\beta > 0$. Then for every $t \geq 0$*

$$\begin{aligned} \mathbb{E} \sup_{s \leq t} |Y_s|^p &\leq C_p t & \text{if } p < 1, \\ \mathbb{E} \sup_{s \leq t} |X_s - s \mathbb{E} X_1|^p &\leq C_p t & \text{if } 1 \leq p \leq 2 \end{aligned}$$

where $Y_t = X_t - t(a - \int_{\{|x| \leq 1\}} x d\nu(x))$. Furthermore, for every $p > 2$

$$\mathbb{E} \sup_{s \leq t} |X_s|^p = O(t) \quad \text{as } t \rightarrow 0$$

for a finite real constant C_p .

If X_1 is symmetric one observes that $Y = X$ since the symmetry of X_1 implies $a = 0$ and the symmetry of ν (see [7]). We emphasize that in view of the Kolmogorov criterion for continuous modifications the above bounds are best possible as concerns powers of t . In case $p > \beta$ and $p \leq 2$, these estimates are due to Millar [6]. However, the Laplace-transform approach in [6] does not work for $p > 2$. Our proof is based on the Burkholder-Davis-Gundy inequality.

For the case $p < \beta$ we need some assumptions on X . Recall that a measurable function $\varphi: (0, c] \rightarrow (0, \infty)$ ($c > 0$) is said to be regularly varying at zero with index $b \in \mathbb{R}$ if, for every $t > 0$,

$$\lim_{x \rightarrow 0} \frac{\varphi(tx)}{\varphi(x)} = t^b.$$

This means that $\varphi(1/x)$ is regularly varying at infinity with index $-b$. Slow variation corresponds to $b = 0$.

Theorem 2 *Let $(X_t)_{t \geq 0}$ be a Lévy process with characteristics $(a, 0, \nu)$ and index β such that $\beta > 0$ and $\mathbb{E}|X_1|^p < +\infty$ for some $p \in (0, \beta)$. Assume that the Lévy measure satisfies*

$$\exists c \in (0, 1], \mathbf{1}_{\{0 < |x| \leq c\}} \nu(dx) \leq \varphi(|x|) \mathbf{1}_{\{0 < |x| \leq c\}} dx \quad (1.5)$$

where $\varphi: (0, c] \rightarrow (0, \infty)$ is a regularly varying function at zero of index $-(\beta + 1)$. Let $l(x) = x^{\beta+1} \varphi(x)$ and assume that $l(1/x)$, $x \geq c$ is locally bounded. Let $\underline{l}(x) = \underline{l}_\beta(x) = l(x^{1/\beta})$.

(a) Assume $\beta > 1$. Then as $t \rightarrow 0$, for every $r \in (\beta, 2]$, $q \in [p \vee 1, \beta)$,

$$\mathbb{E} \sup_{s \leq t} |X_s|^p = O(t^{p/\beta} [\underline{l}(t)^{p/r} + \underline{l}(t)^{p/q}]) \quad \text{if } \beta < 2,$$

$$\mathbb{E} \sup_{s \leq t} |X_s|^p = O(t^{p/\beta} [1 + \underline{l}(t)^{p/q}]) \quad \text{if } \beta = 2.$$

If ν is symmetric then this holds for every $q \in [p, \beta)$.

(b) Assume $\beta < 1$. Then as $t \rightarrow 0$, for every $r \in (\beta, 1]$, $q \in [p, \beta)$

$$\mathbb{E} \sup_{s \leq t} |Y_s|^p = O(t^{p/\beta} [\underline{l}(t)^{p/r} + \underline{l}(t)^{p/q}])$$

where $Y_t = X_t - t(a - \int_{\{|x| \leq 1\}} x d\nu(x))$. If ν is symmetric this holds for every $r \in (\beta, 2]$.

(c) Assume $\beta = 1$ and ν is symmetric. Then as $t \rightarrow 0$, for every $r \in (\beta, 2]$, $q \in [p, \beta)$

$$\mathbb{E} \sup_{s \leq t} |X_s - as|^p = O(t^{p/\beta} [\underline{l}(t)^{p/r} + \underline{l}(t)^{p/q}]).$$

It can be seen from strictly α -stable Lévy processes where $\beta = \alpha$ that the above estimates are best possible as concerns powers of t .

Observe that condition (1.5) is satisfied for a broad class of Lévy processes. It implies that the tail function $t \mapsto \underline{\nu}(t) := \nu([-t, t]^c)$, $t > 0$ of the Lévy measure is dominated, for $t \leq c$, by $2 \int_t^c \varphi(x) ds + \nu(|x| > c)$, a regularly varying function at zero with index $-\beta$, so that $\underline{\nu}(t) = O(t\varphi(t))$ as $t \rightarrow 0$.

Important special cases are as follows.

Corollary 1.1 Assume the situation of Theorem 2 (with ν symmetric if $\beta = 1$) and let U denote any of the processes $X, Y, (X_t - at)_{t \geq 0}$.

(a) Assume that the slowly varying part l of φ is decreasing and unbounded on $(0, c]$ (e.g. $(-\log x)^a$, $a > 0$). Then as $t \rightarrow 0$, for every $\varepsilon \in (0, \beta)$,

$$\mathbb{E} \sup_{s \leq t} |U_s|^p = O(t^{p/\beta} \underline{l}(t)^{p/(\beta-\varepsilon)}).$$

(b) Assume that l is increasing on $(0, c]$ satisfying $l(0+) = 0$ (e.g. $(-\log x)^{-a}$, $a > 0, c < 1$) and $\beta \in (0, 2)$. Then as $t \rightarrow 0$, for every $\varepsilon > 0$,

$$\mathbb{E} \sup_{s \leq t} |U_s|^p = O(t^{p/\beta} \underline{l}(t)^{p/(\beta+\varepsilon)}).$$

The remaining cases $p = \beta \in (0, 2)$ if $\beta \neq 1$ and $p \leq 1$ if $\beta = 1$ are solved under the assumption that the slowly varying part of the function φ in (1.5) is constant.

Theorem 3 Let $(X_t)_{t \geq 0}$ be a Lévy process with characteristics $(a, 0, \nu)$ and index β such that $\beta \in (0, 2)$ and $\mathbb{E} |X_1|^\beta < +\infty$ if $\beta \neq 1$ and $\mathbb{E} |X_1|^p < +\infty$ for some $p \leq 1$ if $\beta = 1$. Assume that the Lévy measure satisfies

$$\exists c \in (0, 1], \exists C \in (0, \infty), \mathbf{1}_{\{0 < |x| \leq c\}} \nu(dx) \leq \frac{C}{|x|^{\beta+1}} \mathbf{1}_{\{0 < |x| \leq c\}} dx. \quad (1.6)$$

Then as $t \rightarrow 0$

$$\mathbb{E} \sup_{s \leq t} |X_s|^\beta = O(t(-\log t)) \quad \text{if } \beta > 1,$$

$$\mathbb{E} \sup_{s \leq t} |Y_s|^\beta = O(t(-\log t)) \quad \text{if } \beta < 1$$

and

$$\mathbb{E} \sup_{s \leq t} |X_s|^p = O((t(-\log t))^p) \quad \text{if } \beta = 1, p \leq 1$$

where the process Y is defined as in Theorem 2.

The above estimates are optimal (see Section 3).

The paper is organized as follows. Section 2 is devoted to the proofs of Theorems 1, 2 and 3. Section 3 contains a collection of examples.

2 Proofs

We will extensively use the following compensation formula (see e.g. [4])

$$\mathbb{E} \int_0^t \int f(s, x) \mu(ds, dx) = \mathbb{E} \sum_{s \leq t} f(s, \Delta X_s) \mathbf{1}_{\{\Delta X_s \neq 0\}} = \int_0^t \int f(s, x) d\nu(x) ds$$

where $f : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}_+$ is a Borel function.

Proof of Theorem 1. Since $\mathbb{E}|X_1|^p < +\infty$ and $p > \beta$ (or $p = \beta$ provided $\int_{\{|x| \leq 1\}} |x|^\beta d\nu(x) < +\infty$ and $\beta > 0$), it follows from (1.2) that

$$\int |x|^p d\nu(x) < +\infty.$$

CASE 1: $0 < p < 1$. In this case we have $\beta < 1$ and hence $\int_{\{|x| \leq 1\}} |x| d\nu(x) < +\infty$. Consequently, X a.s. has finite variation on finite intervals. By (1.4),

$$Y_t = X_t - t \left(a - \int_{\{|x| \leq 1\}} x d\nu(x) \right) = \int_0^t \int x \mu(ds, dx) = \sum_{s \leq t} \Delta X_s$$

so that, using the elementary inequality $(u + v)^p \leq u^p + v^p$,

$$\sup_{s \leq t} |Y_s|^p \leq \left(\sum_{s \leq t} |\Delta X_s| \right)^p \leq \sum_{s \leq t} |\Delta X_s|^p = \int_0^t \int |x|^p \mu(ds, dx).$$

Consequently,

$$\mathbb{E} \sup_{s \leq t} |Y_s|^p \leq t \int |x|^p d\nu(x) \text{ for every } t \geq 0.$$

CASE 2: $1 \leq p \leq 2$. Introduce the martingale

$$M_t := X_t - t\mathbb{E} X_1 = X_t - t \left(a + \int_{\{|x| > 1\}} x d\nu(x) \right) = \int_0^t \int x (\mu - \lambda \otimes \nu)(ds, dx).$$

It follows from the Burkholder-Davis-Gundy inequality (see [5]) that

$$\mathbb{E} \sup_{s \leq t} |M_s|^p \leq C \mathbb{E} [M]_t^{p/2}$$

for some finite constant C . Since $p/2 \leq 1$, the quadratic variation $[M]$ of M satisfies

$$[M]_t^{p/2} = \left(\sum_{s \leq t} |\Delta X_s|^2 \right)^{p/2} \leq \sum_{s \leq t} |\Delta X_s|^p$$

so that

$$\mathbb{E} \sup_{s \leq t} |M_s|^p \leq Ct \int |x|^p d\nu(x) \text{ for every } t \geq 0.$$

CASE 3: $p > 2$. One considers again the martingale Lévy process $M_t = X_t - t\mathbb{E} X_1$. For $k \geq 1$ such that $2^k \leq p$, introduce the martingales

$$N_t^{(k)} := \int_0^t \int |x|^{2^k} (\mu - \lambda \otimes \nu)(ds, dx) = \sum_{s \leq t} |\Delta X_s|^{2^k} - t \int |x|^{2^k} d\nu(x).$$

Set $m := \max\{k \geq 1 : 2^k < p\}$. Again by the Burkholder-Davis-Gundy inequality

$$\begin{aligned}
\mathbb{E} \sup_{s \leq t} |M_s|^p &\leq C \mathbb{E} [M_t]^{p/2} \\
&= C \mathbb{E} \left(t \int x^2 d\nu(x) + N_t^{(1)} \right)^{p/2} \\
&\leq C \left(t^{p/2} \left(\int x^2 d\nu(x) \right)^{p/2} + \mathbb{E} |N_t^{(1)}|^{p/2} \right) \\
&\leq C (t + \mathbb{E} |N_t^{(1)}|^{p/2})
\end{aligned}$$

for every $t \in [0, 1]$ where C is a finite constant that may vary from line to line. Applying successively the Burkholder-Davis-Gundy inequality to the martingales $N^{(k)}$ and exponents $p/2^k > 1, 1 \leq k \leq m$, finally yields

$$\mathbb{E} \sup_{s \leq t} |M_s|^p \leq C(t + \mathbb{E} [N^{(m)}]_t^{p/2^{m+1}}) \quad \text{for every } t \in [0, 1].$$

Using $p \leq 2^{m+1}$, one gets

$$[N^{(m)}]_t^{p/2^{m+1}} = \left(\sum_{s \leq t} |\Delta X_s|^{2^{m+1}} \right)^{p/2^{m+1}} \leq \sum_{s \leq t} |\Delta X_s|^p$$

so that

$$\mathbb{E} \sup_{s \leq t} |M_s|^p \leq C \left(t + t \int |x|^p d\nu(x) \right) \quad \text{for every } t \in [0, 1].$$

This implies $\mathbb{E} \sup_{s \leq t} |X_s|^p = O(t)$ as $t \rightarrow 0$. □

Proof of Theorems 2 and 3. Let $p \leq \beta$ and fix $c \in (0, 1]$. Let $\nu_1 = \mathbf{1}_{\{|x| \leq c\}} \cdot \nu$ and $\nu_2 = \mathbf{1}_{\{|x| > c\}} \cdot \nu$. Construct Lévy processes $X^{(1)}$ and $X^{(2)}$ such that $X \stackrel{d}{=} X^{(1)} + X^{(2)}$ and $X^{(2)}$ is a compound Poisson process with Lévy measure ν_2 . Then $\beta = \beta(X) = \beta(X^{(1)}), \beta(X^{(2)}) = 0$, $\mathbb{E} |X^{(1)}|^q < +\infty$ for every $q > 0$ and $\mathbb{E} |X_1^{(2)}|^p < +\infty$. It follows e.g. from Theorem 1 that for every $t \geq 0$,

$$\mathbb{E} \sup_{s \leq t} |X_s^{(2)}|^p \leq C_p t \quad \text{if } p < 1, \tag{2.1}$$

$$\mathbb{E} \sup_{s \leq t} |X^{(2)} - s \mathbb{E} X_1^{(2)}|^p \leq C_p t \quad \text{if } 1 \leq p \leq 2$$

where $\mathbb{E} X_1^{(2)} = \int x d\nu_2(x) = \int_{\{|x| > c\}} x d\nu(x)$.

As concerns $X^{(1)}$, consider the martingale

$$Z_t^{(1)} := X_t^{(1)} - t \mathbb{E} X_1^{(1)} = X_t^{(1)} - t \left(a - \int x \mathbf{1}_{\{c < |x| \leq 1\}} d\nu(x) \right) = \int_0^t \int x (\mu_1 - \lambda \otimes \nu_1)(ds, dx)$$

where μ_1 denotes the Poisson random measure associated with the jumps of $X^{(1)}$. The starting idea is to part the 'small' and the 'big' jumps of $X^{(1)}$ in a non homogeneous way with respect to the function $s \mapsto s^{1/\beta}$. Indeed one may decompose $Z^{(1)}$ as follows

$$Z^{(1)} = M + N$$

where

$$M_t := \int_0^t \int x \mathbf{1}_{\{|x| \leq s^{1/\beta}\}} (\mu_1 - \lambda \otimes \nu_1)(ds, dx)$$

and

$$N_t := \int_0^t \int x \mathbf{1}_{\{|x| > s^{1/\beta}\}} (\mu_1 - \lambda \otimes \nu_1)(ds, dx)$$

are martingales. Observe that for every $q > 0$ and $t \geq 0$,

$$\begin{aligned} \int_0^t \int |x|^q \mathbf{1}_{\{|x| > s^{1/\beta}\}} d\nu_1(x) ds &= \int |x|^q (|x|^\beta \wedge t) d\nu_1(x) \\ &\leq \int_{\{|x| \leq c\}} |x|^{\beta+q} d\nu(x) < +\infty. \end{aligned}$$

Consequently,

$$N_t = \int_0^t \int x \mathbf{1}_{\{|x| > s^{1/\beta}\}} d\mu_1(s, x) - \psi(t)$$

where $\psi(t) := \int_0^t \int x \mathbf{1}_{\{|x| > s^{1/\beta}\}} d\nu_1(x) ds$. Furthermore, for every $r > \beta$ or $r = 2$ and $t \geq 0$

$$\int_0^t \int |x|^r \mathbf{1}_{\{|x| \leq s^{1/\beta}\}} d\nu_1(x) ds \leq t \int_{\{|x| \leq c\}} |x|^r d\nu(x) < +\infty. \quad (2.2)$$

In the sequel let C denote a finite constant that may vary from line to line.

We first claim that for every $t \geq 0, r \in (\beta, 2] \cap [1, 2]$ and for $r = 2$,

$$\mathbb{E} \sup_{s \leq t} |M_s| \leq C \left(\int_0^t \int |x|^r \mathbf{1}_{\{|x| \leq s^{1/\beta}\}} d\nu_1(x) ds \right)^{p/r}. \quad (2.3)$$

In fact, it follows from the Burkholder-Davis-Gundy inequality and from $p/r \leq 1, r/2 \leq 1$ that

$$\begin{aligned} \mathbb{E} \sup_{s \leq t} |M_s|^p &\leq \left(\mathbb{E} \sup_{s \leq t} |M_s|^r \right)^{p/r} \\ &\leq C \left(\mathbb{E} [M]_t^{r/2} \right)^{p/r} \\ &= C \left(\mathbb{E} \left(\sum_{s \leq t} |\Delta X_s^{(1)}|^2 \mathbf{1}_{\{|\Delta X_s^{(1)}| \leq s^{1/\beta}\}} \right)^{r/2} \right)^{p/r} \\ &\leq C \left(\mathbb{E} \sum_{s \leq t} |\Delta X_s^{(1)}|^r \mathbf{1}_{\{|\Delta X_s^{(1)}| \leq s^{1/\beta}\}} \right)^{p/r} \\ &= C \left(\int_0^t \int |x|^r \mathbf{1}_{\{|x| \leq s^{1/\beta}\}} d\nu_1(x) ds \right)^{p/r}. \end{aligned}$$

Exactly as for M , one gets for every $t \geq 0$ and every $q \in [p, 2] \cap [1, 2]$ that

$$\mathbb{E} \sup_{s \leq t} |N_s|^p \leq C \left(\int_0^t \int |x|^q \mathbf{1}_{\{|x| > s^{1/\beta}\}} d\nu_1(x) ds \right)^{p/q}. \quad (2.4)$$

If ν is symmetric then (2.4) holds for every $q \in [p, 2]$ (which of course provides additional information in case $p < 1$ only). Indeed, $\psi = 0$ by the symmetry of ν so that

$$N_t = \int_0^t \int x \mathbf{1}_{\{|x| > s^{1/\beta}\}} d\mu_1(s, x)$$

and for $q \in [p, 1]$

$$\begin{aligned}
\mathbb{E} \sup_{s \leq t} \left| \int_0^s \int x \mathbf{1}_{\{|x| > u^{1/\beta}\}} \mu_1(du, dx) \right|^p &\leq \left(\mathbb{E} \sup_{s \leq t} \left| \int_0^t \int x \mathbf{1}_{\{|x| > u^{1/\beta}\}} \mu_1(du, dx) \right|^q \right)^{p/q} \\
&\leq \left(\mathbb{E} \sum_{s \leq t} \left| \Delta X_s^{(1)} \right|^q \mathbf{1}_{\{|\Delta X_s^{(1)}| > s^{1/\beta}\}} \right)^{p/q} \\
&= \left(\int_0^t \int |x|^q \mathbf{1}_{\{|x| > s^{1/\beta}\}} d\nu_1(x) ds \right)^{p/q}.
\end{aligned} \tag{2.5}$$

In the case $\beta < 1$ we consider the process

$$\begin{aligned}
Y_t^{(1)} &:= Z_t^{(1)} + t \int x d\nu_1(x) = X_t^{(1)} - t \left(a - \int_{\{|x| \leq 1\}} x d\nu(x) \right) \\
&= M_t + N_t + t \int x d\nu_1(x) \\
&= \int_0^t \int x \mathbf{1}_{\{|x| \leq s^{1/\beta}\}} \mu_1(ds, dx) + \int_0^1 \int x \mathbf{1}_{\{|x| > s^{1/\beta}\}} \mu_1(ds, dx).
\end{aligned}$$

Exactly as in (2.5) one shows that for $t \geq 0$ and $r \in (\beta, 1]$

$$\mathbb{E} \sup_{s \leq t} \left| \int_0^s \int x \mathbf{1}_{\{|x| \leq u^{1/\beta}\}} \mu_1(du, dx) \right|^p \leq \left(\int_0^t \int |x|^r \mathbf{1}_{\{|x| \leq s^{1/\beta}\}} d\nu_1(x) ds \right)^{p/r}. \tag{2.6}$$

Combining (2.1) and (2.3) - (2.6) we obtain the following estimates. Let

$$Z_t = X_t - t \left(a - \int x \mathbf{1}_{\{|x| \leq 1\}} d\nu(x) \right).$$

CASE 1: $\beta \geq 1$ and $p < 1$. Then for every $t \geq 0, r \in (\beta, 2] \cup \{2\}, q \in [1, 2]$,

$$\begin{aligned}
\mathbb{E} \sup_{s \leq t} |Z_s|^p &\leq C \left(t + \left(\int_0^t \int |x|^r \mathbf{1}_{\{|x| \leq s^{1/\beta}\}} d\nu_1(x) ds \right)^{p/r} \right. \\
&\quad \left. + \left(\int_0^t \int |x|^q \mathbf{1}_{\{|x| > s^{1/\beta}\}} d\nu_1(x) ds \right)^{p/q} \right).
\end{aligned} \tag{2.7}$$

If ν is symmetric (2.7) is even valid for every $q \in [p, 2]$.

CASE 2: $\beta \geq 1$ and $p \geq 1$. Then for every $t \geq 0, r \in (\beta, 2] \cup \{2\}, q \in [p, 2]$,

$$\begin{aligned}
\mathbb{E} \sup_{s \leq t} |X_s - s \mathbb{E} X_1|^p &\leq C \left(t + \left(\int_0^t \int |x|^r \mathbf{1}_{\{|x| \leq s^{1/\beta}\}} d\nu_1(x) ds \right)^{p/r} \right. \\
&\quad \left. + \left(\int_0^t \int |x|^q \mathbf{1}_{\{|x| > s^{1/\beta}\}} d\nu_1(x) ds \right)^{p/q} \right).
\end{aligned} \tag{2.8}$$

CASE 3: $\beta < 1$. Then for every $t \geq 0, r \in (\beta, 1], q \in [p, 1]$

$$\begin{aligned}
\mathbb{E} \sup_{s \leq t} |Y_s|^p &\leq C \left(t + \left(\int_0^t \int |x|^r \mathbf{1}_{\{|x| \leq s^{1/\beta}\}} d\nu_1(x) ds \right)^{p/r} \right. \\
&\quad \left. + \left(\int_0^t \int |x|^q \mathbf{1}_{\{|x| > s^{1/\beta}\}} d\nu_1(x) ds \right)^{p/q} \right).
\end{aligned} \tag{2.9}$$

If ν is symmetric then $Y = Z = (X_t - at)_{t \geq 0}$ and (2.9) is valid for every $r \in (\beta, 2], q \in [p, 2]$.

Now we deduce Theorem 2. Assume $p \in (0, \beta)$ and (1.5). The constant c in the above decomposition of X is specified by the constant from (1.5). Then one just needs to investigate the integrals appearing in the right hand side of the inequalities (2.7) - (2.10). One observes that Theorem 1.5.11 in [1] yields for $r > \beta$,

$$\int |x|^r \varphi(|x|) \mathbf{1}_{\{|x| \leq s^{1/\beta}\}} dx \sim \frac{2}{r - \beta} s^{\frac{r}{\beta} - 1} l(s^{1/\beta}) \text{ as } s \rightarrow 0$$

which in turn implies that for small t ,

$$\begin{aligned} \int_0^t \int |x|^r \mathbf{1}_{\{|x| \leq s^{1/\beta}\}} d\nu_1(x) ds &\leq \int_0^t \int |x|^r \varphi(|x|) \mathbf{1}_{\{|x| \leq s^{1/\beta}\}} dx ds \\ &\sim \frac{2\beta}{(r - \beta)r} t^{r/\beta} l(t^{1/\beta}) \text{ as } t \rightarrow 0. \end{aligned} \quad (2.10)$$

Similarly, for $0 < q < \beta$,

$$\begin{aligned} \int_0^t \int |x|^q \mathbf{1}_{\{|x| > s^{1/\beta}\}} d\nu_1(x) ds &\leq \int_0^t \int |x|^q \varphi(|x|) \mathbf{1}_{\{|x| > s^{1/\beta}\}} dx ds \\ &\sim \frac{2\beta}{(\beta - q)q} t^{q/\beta} l(t^{1/\beta}) \text{ as } t \rightarrow 0. \end{aligned} \quad (2.11)$$

Using (2.2) for the case $\beta = 2$ and $t + t^p = o(t^{p/\beta} l(t)^\alpha)$ as $t \rightarrow 0, \alpha > 0$, for the case $\beta > 1$ one derives Theorem 2.

As for Theorem 3, one just needs a suitable choice of q in (2.7) - (2.9). Note that by (1.6) for every $\beta \in (0, 2)$ and $t \leq c^\beta$,

$$\begin{aligned} \int_0^t \int |x|^\beta \mathbf{1}_{\{|x| > s^{1/\beta}\}} d\nu_1(x) ds &= \int_0^t \int |x|^\beta \mathbf{1}_{\{c \geq |x| > s^{1/\beta}\}} d\nu(x) ds \\ &\leq C \int_0^t \int |x|^{-1} \mathbf{1}_{\{c \geq |x| > s^{1/\beta}\}} dx ds \\ &= Ct(-\log t) \end{aligned}$$

so that $q = \beta$ is the right choice. (This choice of q is optimal.) Since by (2.10), for $r \in (\beta, 2] (\neq \emptyset)$,

$$\int_0^t \int |x|^r \mathbf{1}_{\{|x| \leq s^{1/\beta}\}} d\nu_1(x) ds = O(t^{r/\beta})$$

the assertions follow from (2.7) - (2.9). \square

3 Examples

Let K_ν denote the modified Bessel function of the third kind and index ν given by

$$K_\nu(z) = \frac{1}{2} \int_0^\infty u^{\nu-1} \exp\left(-\frac{z}{2}\left(u + \frac{1}{u}\right)\right) du, \quad z > 0.$$

• *The Γ -process* is a subordinator (increasing Lévy process) whose distribution \mathbb{P}_{X_t} at time $t > 0$ is a $\Gamma(1, t)$ -distribution

$$\mathbb{P}_{X_t}(dx) = \frac{1}{\Gamma(t)} x^{t-1} e^{-x} \mathbf{1}_{(0, \infty)}(x) dx.$$

The characteristics are given by

$$\nu(dx) = \frac{1}{x} e^{-x} \mathbf{1}_{(0,\infty)}(x) dx$$

and $a = \int_0^1 x d\nu(x) = 1 - e^{-1}$ so that $\beta = 0$ and $Y = X$. It follows from Theorem 1 that

$$\mathbb{E} \sup_{s \leq t} X_s^p = \mathbb{E} X_t^p = O(t)$$

for every $p > 0$. This is clearly the true rate since

$$\mathbb{E} X_t^p = \frac{\Gamma(p+t)}{\Gamma(t+1)} t \sim \Gamma(p)t \text{ as } t \rightarrow 0.$$

- The α -stable Lévy Processes indexed by $\alpha \in (0, 2)$ have Lévy measure

$$\nu(dx) = \left(\frac{C_1}{x^{\alpha+1}} \mathbf{1}_{(0,\infty)}(x) + \frac{C_2}{|x|^{\alpha+1}} \mathbf{1}_{(-\infty,0)}(x) \right) dx$$

with $C_i \geq 0, C_1 + C_2 > 0$ so that $\mathbb{E} |X_1|^p < +\infty$ for $p \in (0, \alpha)$, $\mathbb{E} |X_1|^\alpha = \infty$ and $\beta = \alpha$. It follows from Theorems 2 and 3 that for $p \in (0, \alpha)$,

$$\begin{aligned} \mathbb{E} \sup_{s \leq t} |X_s|^p &= O(t^{p/\alpha}) & \text{if } \alpha > 1, \\ \mathbb{E} \sup_{s \leq t} |Y_s|^p &= O(t^{p/\alpha}) & \text{if } \alpha < 1, \\ \mathbb{E} \sup_{s \leq t} |X_s|^p &= O((t(-\log t))^p) & \text{if } \alpha = 1. \end{aligned}$$

Here Theorem 3 gives the true rate provided X is not strictly stable. In fact, if $\alpha = 1$ the scaling property in this case says that $X_t \stackrel{d}{=} tX_1 + Ct \log t$ for some real constant $C \neq 0$ (see [7], p.87) so that for $p < 1$

$$\mathbb{E} |X_t|^p = t^p \mathbb{E} |X_1 + C \log t|^p \sim C^p t^p |\log t|^p \quad \text{as } t \rightarrow 0.$$

Now assume that X is *strictly* α -stable. If $\alpha < 1$, then $a = \int_{|x| \leq 1} x d\nu(x)$ and thus $Y = X$ and if $\alpha = 1$, then ν is symmetric (see [7]). Consequently, by Theorem 2, for every $\alpha \in (0, 2), p \in (0, \alpha)$,

$$\mathbb{E} \sup_{s \leq t} |X_s|^p = O(t^{p/\alpha}).$$

In this case Theorem 2 provides the true rate since the self-similarity property of strictly stable Lévy processes implies

$$\mathbb{E} \sup_{s \leq t} |X_s|^p = t^{p/\alpha} \mathbb{E} \sup_{s \leq 1} |X_s|^p.$$

- *Tempered stable processes* are subordinators with Lévy measure

$$\nu(dx) = \frac{2^\alpha \cdot \alpha}{\Gamma(1-\alpha)} x^{-(\alpha+1)} \exp\left(-\frac{1}{2} \gamma^{1/\alpha} x\right) \mathbf{1}_{(0,\infty)}(x) dx$$

and first characteristic $a = \int_0^1 x d\nu(x), \alpha \in (0, 1), \gamma > 0$ (see [8]) so that $\beta = \alpha, Y = X$ and $\mathbb{E} X_1^p < +\infty$ for every $p > 0$. The distribution of X_t is not generally known. It follows from Theorems 1,2 and 3 that

$$\begin{aligned} \mathbb{E} X_t^p &= O(t) & \text{if } p > \alpha, \\ \mathbb{E} X_t^p &= O(t^{p/\alpha}) & \text{if } p < \alpha \\ \mathbb{E} X_t^\alpha &= O(t(-\log t)) & \text{if } p = \alpha. \end{aligned}$$

For $\alpha = 1/2$, the process reduces to the *inverse Gaussian process* whose distribution \mathbb{P}_{X_t} at time $t > 0$ is given by

$$\mathbb{P}_{X_t}(dx) = \frac{t}{\sqrt{2\pi}} x^{-3/2} \exp\left(-\frac{1}{2}\left(\frac{t}{\sqrt{x}} - \gamma\sqrt{x}\right)^2\right) \mathbf{1}_{(0,\infty)}(x) dx.$$

In this case all rates are the true rates. In fact, for $p > 0$,

$$\begin{aligned} \mathbb{E}X_t^p &= \frac{t}{\sqrt{2\pi}} e^{t\gamma} \int_0^\infty x^{p-3/2} \exp\left(-\frac{1}{2}\left(\frac{t}{\sqrt{x}} + \gamma\sqrt{x}\right)^2\right) dx \\ &= \frac{t}{\sqrt{2\pi}} e^{t\gamma} \left(\frac{1}{\gamma}\right)^{p-3/2} t^{p-1/2} \int_0^\infty y^{p-3/2} \exp\left(-\frac{t\gamma}{2}\left(\frac{1}{y} + \gamma^2 y\right)\right) dy \\ &= \frac{2}{\sqrt{2\pi}} \gamma^{p-3/2} t^{p+1/2} e^{t\gamma} K_{p-1/2}(t\gamma) \end{aligned}$$

and, as $z \rightarrow 0$,

$$\begin{aligned} K_{p-1/2}(z) &\sim \frac{C_p}{z^{p-1/2}} & \text{if } p > \frac{1}{2}, \\ K_{p-1/2}(z) &\sim \frac{C_p}{z^{1/2-p}} & \text{if } p < \frac{1}{2} \\ K_0(z) &\sim |\log z| \end{aligned}$$

where $C_p = 2^{p-3/2}\Gamma(p-1/2)$ if $p > 1/2$ and $C_p = 2^{-p-1/2}\Gamma(\frac{1}{2}-p)$ if $p < 1/2$.

• *The Normal Inverse Gaussian (NIG)* process was introduced by Barndorff-Nielsen and has been used in financial modeling (see [8]). The NIG process is a Lévy process with characteristics $(a, 0, \nu)$ where

$$\begin{aligned} \nu(dx) &= \frac{\delta\alpha}{\pi} \frac{\exp(\gamma x) K_1(\alpha|x|)}{|x|} dx, \\ a &= \frac{2\delta\alpha}{\pi} \int_0^1 \sinh(\gamma x) K_1(\alpha x) dx, \end{aligned}$$

$\alpha > 0$, $\gamma \in (-\alpha, \alpha)$, $\delta > 0$. Since $K_1(|z|) \sim |z|^{-1}$ as $z \rightarrow 0$, the Lévy density behaves like $\delta\pi^{-1}|x|^{-2}$ as $x \rightarrow 0$ so that (1.6) is satisfied with $\beta = 1$. One also checks that $\mathbb{E}|X_1|^p < +\infty$ for every $p > 0$. It follows from Theorems 1 and 3 that, as $t \rightarrow 0$

$$\begin{aligned} \mathbb{E} \sup_{s \leq t} |X_s|^p &= O(t) & \text{if } p > 1, \\ \mathbb{E} \sup_{s \leq t} |X_s|^p &= O((t(-\log t))^p) & \text{if } p \leq 1. \end{aligned}$$

If $\gamma = 0$, then ν is symmetric and by Theorem 2,

$$\mathbb{E} \sup_{s \leq t} |X_s|^p = O(t^p) \quad \text{if } p < 1.$$

The distribution \mathbb{P}_{X_t} at time $t > 0$ is given by

$$\mathbb{P}_{X_t}(dx) = \frac{t\delta\alpha}{\pi} \exp(t\delta\sqrt{\alpha^2 - \gamma^2} + \gamma x) \frac{K_1(\alpha\sqrt{t^2\delta^2 + x^2})}{\sqrt{t^2\delta^2 + x^2}} dx$$

so that Theorem 3 gives the true rate for $p = \beta = 1$ in the symmetric case. In fact, assuming $\gamma = 0$, we get as $t \rightarrow 0$

$$\mathbb{E}|X_t| = \frac{2t\delta\alpha}{\pi} e^{t\delta\alpha} \int_0^\infty \frac{x K_1(\alpha\sqrt{t^2\delta^2 + x^2})}{\sqrt{t^2\delta^2 + x^2}} dx$$

$$\begin{aligned}
&= \frac{2t\delta\alpha}{\pi} e^{t\delta\alpha} \int_{t\delta}^{\infty} K_1(\alpha y) dy \\
&\sim \frac{2\delta}{\pi} t \int_{t\delta}^1 \frac{1}{y} dy \\
&\sim \frac{2\delta}{\pi} t (-\log(t)).
\end{aligned}$$

• *Hyperbolic Lévy motions* have been applied to option pricing in finance (see [3]). These processes are Lévy processes whose distribution \mathbb{P}_{X_1} at time $t = 1$ is a symmetric (centered) hyperbolic distribution

$$\mathbb{P}_{X_1}(dx) = C \exp(-\delta \sqrt{1 + (x/\gamma)^2}) dx, \quad \gamma, \delta > 0.$$

Hyperbolic Lévy processes have characteristics $(0, 0, \nu)$ and satisfy $\mathbb{E}|X_1|^p < +\infty$ for every $p > 0$. In particular, they are martingales. There (rather involved) symmetric Lévy measure has a Lebesgue density that behaves like Cx^{-2} as $x \rightarrow 0$ so that (1.6) is satisfied with $\beta = 1$. Consequently, by Theorems 1, 2 and 3, as $t \rightarrow 0$

$$\begin{aligned}
\mathbb{E} \sup_{s \leq t} |X_s|^p &= O(t) \quad \text{if } p > 1, \\
\mathbb{E} \sup_{s \leq t} |X_s|^p &= O(t^p) \quad \text{if } p < 1, \\
\mathbb{E} \sup_{s \leq t} |X_s| &= O(t(-\log t)) \quad \text{if } p = 1.
\end{aligned}$$

• *Meixner processes* are Lévy processes without Brownian component and with Lévy measure given by

$$\nu(dx) = \frac{\delta e^{\gamma x}}{x \sinh(\pi x)} dx, \quad \delta > 0, \quad \gamma \in (-\pi, \pi)$$

(see [8]). The density behaves like $\delta/\pi x^2$ as $x \rightarrow 0$ so that (1.6) is satisfied with $\beta = 1$. Using (1.2) one observes that $\mathbb{E}|X_1|^p < +\infty$ for every $p > 0$. It follows from Theorems 1 and 3 that

$$\begin{aligned}
\mathbb{E} \sup_{s \leq t} |X_s|^p &= O(t) \quad \text{if } p > 1, \\
\mathbb{E} \sup_{s \leq t} |X_s|^p &= O((t(-\log t))^p) \quad \text{if } p \leq 1.
\end{aligned}$$

If $\gamma = 0$, then ν is symmetric and hence Theorem 2 yields

$$\mathbb{E} \sup_{s \leq t} |X_s|^p = O(t^p) \quad \text{if } p < 1.$$

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